RuBy: a bipolar-valued outranking method for the best choice decision problem

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Abstract The main concern of this article is a detailed presentation of the RuBy methodology for the best choice problem in the context of multiple criteria decision aid. We focus more particularly on pairwise comparisons of decision objects which lead to the concept of bipolar-valued outranking digraph. The work is centred around a list of five pragmatic principles which are required in the context of a best choice problematique¹. Their thorough study and implementation in the outranking digraph leads us to define a best choice recommendation as an extension of the classical kernel concept. Finally, we present and discuss the resolution algorithm of the RuBy best choice method.

Key words: Best choice, bipolar-valued outranking relation, hyperkernel.

Introduction

The goal of this article is to discuss how to rationally determine the best choice (denoted BC) among a set of potential decision objects (or alternatives) in the context of multiple criteria decision aid. The starting point of the discussion is a so-called bipolar-valued outranking relation defined on the finite set X of alternatives. Such an outranking relation expresses the

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¹ We use the French word here because the English language lacks a suitable term (see also (BS03)).
degree of confidence in the truth of a global pairwise preference situation between the alternatives, combining an at least as good as statement with the absence of any local veto. This work is motivated by the fact that this decision aid problem is generally non trivial. Indeed, the outranking relation results from a multidimensional preference aggregation, involving a majority concordance principle, which in general does not produce a total or transitive relation.

From a pragmatic point of view, the goal of the BC problem is to select a unique ultimate best alternative. We situate ourselves in the context of a progressive and interactive approach 2.1 and therefore, this type of decision aid consists in a first step in the elicitation of a subset of good alternatives which is as restricted as possible. It is meant to help the decision maker (DM) to get as close as possible to the selection of a unique best alternative. In case this first recommendation consists of several candidates, the decision aiding process is restarted with new and more detailed information in order to assist the DM in selecting the final best alternative.

Apart from the European multiple criteria decision aid community (Roy85), this specific BC problem has attracted quite low attention by the Operational Research (OR) field. Seminal work on it goes back to the first articles of Roy on the ELECTRE I methods (Roy68; Roy69). After Kitainik (Kit93), interest in solving the BC problem differently from the classical optimisation paradigm has reappeared. The recent work of Bisdorff and Roubens on valued kernels (BR96) has resulted in new attempts to solve the BCP directly from the valued outranking digraph. After first positive results (Bis00), methodological difficulties appeared when applying the outranking kernel concept to highly non transitive and partial outranking relations.

In this paper, we therefore suggest a new proposal to solve the BC problem and to revisit the logical and pragmatic foundations of this problematique. The objective is to suggest a new and innovative decision aid methodology in the tradition of the pioneering work of Roy and Bouyssou (RB93).

The paper is organised as follows. In Section 1, we introduce the basic concepts and notations which are necessary for our future discourse. Then, in Section 2, we revisit the very foundations of the BC problem, briefly present the classical ELECTRE approach, and list the new principles which could be required for the BC problematique. The third Section deals with the translation of these principles into properties in the bipolar-valued outranking digraph and presents the concept of hyperkernel, an extension of the classical kernel of a digraph. In the fourth and last section, we show how to determine these hyperkernels and detail the algorithm for the RuBy BC method.

Historically, the BC decision problem in the context of outranking relations has been solved by the use of the kernel of a digraph. As we will show, this concept might be too restrictive in certain situations. This observation
is one of the main motivations for the use of the RuBY BC method. In order to clearly understand the discourse of Sections 2, 3 and 4 we start our discussion with an introductory section presenting formal definitions of the basic notions used in the present paper.

1 On choices in bipolar-valued outranking digraphs

In order to delimit the framework of our discourse, we introduce in this section the fundamental concepts and notations about about bipolar-valued outranking digraphs and potential best choices. We start by establishing the backbone of the RuBY method, namely the bipolar-valued credibility scale modelling degrees of confidence in the truth of logical statements.

1.1 Bipolar-valued credibility calculus

Let \( \xi \) be a propositional statement like – decision alternative \( a \) is a best choice – or – decision alternative \( a \) is at least as good as decision alternative \( b \). In a decision process, a decision maker may either accept or reject these statements following his degree of confidence in their truth (Bis00). This degree of confidence in the truth of a statement – its credibility – may be represented with the help of a rational credibility scale \( L = [-1, 1] \) supporting the following truth-denotation semantics:

1. Let \( r \in L \) denote the credibility of a statement \( \xi \). If \( r = +1 \) (resp. \( r = -1 \)) then it is assumed that \( \xi \) is certainly true (resp. false). If \( 0 < r < +1 \) (resp. \( -1 < r < 0 \)) then it is assumed that \( \xi \) is more true than false (resp. more false than true). If \( r = 0 \) then \( \xi \) is logically undetermined, i.e. \( \xi \) could be either true or false;
2. Let \( \xi \) and \( \psi \) be two propositional statements to which are associated credibilities \( r \) and \( s \in L \). If \( r > s > 0 \) (resp. \( r < s < 0 \)) then it is assumed that the truth (resp. falsity) of \( \xi \) is more credible than that of \( \psi \);
3. Let \( \xi \) and \( \psi \) be two propositional statements to which are associated credibilities \( r \) and \( s \in L \). The truthfulness of the disjunction \( \xi \lor \psi \) (resp. the conjunction \( \xi \land \psi \)) of these statements corresponds to the maximum of their credibilities: \( \max(r, s) \) (resp. the minimum of their credibilities: \( \min(r, s) \));
4. If \( r \in L \) denotes the degree of confidence in the truth of a propositional statement \( \xi \), then \(-r \in L \) denotes the degree of confidence in its untruth, i.e. the truthfulness of the logical negation of \( \xi \) (\( \neg \xi \)).

Definition 1.1. The credibility degree associated with the truth of a propositional statement \( \xi \) and defined in a credibility domain \( L \) verifying properties (1) to (4) is called a bipolar-valued characterisation of \( \xi \).
A consequence of property (4) is that the graduation of confidence degrees concerns necessarily at the same time the affirmation as well as the negation of a propositional statement (Win84). Starting from $+1$ (certainly true) and $-1$ (certainly false), one can approach the central undetermined degree of credibility 0 by a gradual weakening of the degrees of confidence. This central point in $L$ is a so-called negational fixpoint (Bis00; Bis02).

**Definition 1.2.** The degree of logical determination, i.e. determinateness, $D(\xi)$ of a propositional statement $\xi$ is given by the absolute value of its bipolar-valued characterisation: $D(\xi) = |r|$.

Consequently, for both a certainly true and a certainly false statement, the determinateness is 1. On the opposite, for an undetermined statement, this determinateness equals 0.

This clearly establishes the central degree 0 as an important neutral value in the bipolar credibility calculus. Propositions characterised with this degree 0 may be either seen as suspended or missing statements (Bis02). This situation corresponds to what we call a suspension of judgement. It is a temporary delay in characterising the actual truth or falsity of a propositional statement. In the framework of decision aid, such an undetermined situation may become determined either as true or false in a later stage of the process.

The following section allows us to define the concept of bipolar-valued outranking digraph which is the preferential support for the RUBY best choice decision aiding methodology.

### 1.2 The bipolar-valued outranking digraph

Our starting point is a decision aiding problem on a finite set $X = \{x, y, z, \ldots\}$ of decision objects (or alternatives) evaluated on a finite, coherent family $F = \{1, \ldots, p\}$ of $p$ criteria. To each criterion $j$ of $F$ is associated its significance represented by a rational number $w_j$ from the open interval $]0, 1[$ such that $\sum_{j=1}^{p} w_j = 1$. Besides, to each criterion $j$ is connected a rational (normalised) preference scale in $[0, 1]$ which allows to compare the performances of the decision objects on the corresponding preference dimension.

Let $g_j(x)$ and $g_j(y)$ be the performances of two alternatives $x$ and $y$ of $X$ on criterion $j$. The difference of the performances $g_j(x) - g_j(y)$ is written $\Delta_j(x, y)$. Each preference scale for each criterion $j$ supports a variable rational indifferenece threshold $h_j(g_j(x)) \in [0, 1]$, a weak preference threshold $q_j(g_j(x)) \in [h_j(g_j(x)), 1]$, a weak veto threshold $wv_j(g_j(x)) \in [q_j(g_j(x)), 1] \cup \{2\}$ and a strong veto threshold $v_j(g_j(x)) \in [wv_j(g_j(x)), 1] \cup \{2\}$, where the complete absence of veto is modelled via the value 2.

Classically, an outranking situation $x \succ_y$ between two decision alternatives $x$ and $y$ of $X$ is assumed to hold if there is a sufficient majority of
criteria which supports an “at least as good as” preferential statement and there is no criterion which raises a veto against it (Roy85). As we are going to show, this definition leads quite naturally to a bipolar-valued characterisation of binary outranking statements.

Indeed, in order to characterise a local “at least as good as” situation between two alternatives $x$ and $y$ of $X$ for each criterion $j \in F$, we use the following criterion-function: $C_j : X \times X \rightarrow \{-1, 0, 1\}$ such that:

$$C_j(x, y) = \begin{cases} 
1 & \text{if } \Delta_j(x, y) > -h_j(g_j(x)) ; \\
-1 & \text{if } \Delta_j(x, y) \leq -g_j(g_j(x)) ; \\
0 & \text{otherwise}.
\end{cases}$$

Following the truth-denotation semantics of the bipolar-valued characterisation, determinateness 0 is assigned to $C_j(x, y)$ in case it cannot be determined whether alternative $x$ is at least as good as alternative $y$ or not (see Subsection 1.1).

Similarly, the local veto situation for each criterion $j \in F$ is characterised via a criterion-based veto-function: $V_j : X \times X \rightarrow \{-1, 0, 1\}$ where:

$$V_j(x, y) = \begin{cases} 
1 & \text{if } \Delta_j(x, y) \leq -v_j(g_j(x)) ; \\
-1 & \text{if } \Delta_j(x, y) > -wv_j(g_j(x)) ; \\
0 & \text{otherwise}.
\end{cases}$$

Again, according to the semantics of the bipolar-valued characterisation, the veto function $V_j$ renders a logically undetermined response when the loss in performances between two alternatives lies in between the weak and the strong veto thresholds $wv_j$ and $v_j$.

The global outranking index $\tilde{S}$, defined for all pairs of alternatives $(x, y) \in X \times X$, conjunctively combines a global concordance index, aggregating all local “at least as good as” statements, and the absence of a veto observed on an individual criterion.

$$\tilde{S}(x, y) = \min\{\tilde{C}(x, y), -V_1(x, y), \ldots, -V_p(x, y)\}, \quad (1.1)$$

where the global concordance index $\tilde{C}(x, y)$ is defined as follows:

$$\tilde{C}(x, y) = \sum_{j \in F} (w_j \cdot C_j(x, y)) \quad \forall x, y \in X. \quad (1.2)$$

The min operator in Formula 1.1 translates the conjunction between the global concordance index $\tilde{C}(x, y)$ and the negated criterion-based veto indexes $-V_j(x, y) (\forall j \in F)$. In the case of absence of veto, the resulting outranking index $\tilde{S}$ equals the global concordance index $\tilde{C}$. Following Formulae (1.1) and (1.2), $\tilde{S}$ is a function from $X \times X$ to $\mathcal{L}$ representing the degree of confidence in the truth of the outranking situation observed between each
pair of alternatives. \( \overline{S} \) is called the bipolar-valued characterisation of the outranking situation \( S \), or in short, a *bipolar-valued outranking relation*.

The maximum possible value of the valuation \( \overline{S}(x, y) = +1 \) is reached in the case of unanimous concordance, whereas the minimum value \( \overline{S}(x, y) = -1 \) is obtained either in the case of unanimous discordance, or if we observe a veto situation on some criterion. The median situation 0 represents a case of undeterminateness: either there are neither enough arguments in favour nor against a given outranking statement or, a potentially sufficient majority in favour of the outranking is outbalanced by an undetermined, i.e. a potential veto situation.

We can easily recover the truth-denotation semantics from the previous Subsection (1.1). For any two alternatives \( x \) and \( y \) of \( X \),

- \( \overline{S}(x, y) = +1 \) signifies that the statement "\( x S y \)" is *certainly true*;
- \( \overline{S}(x, y) > 0 \) signifies that statement "\( x S y \)" is *more true than false*. A sufficient majority of criteria warrants the truth of the outranking;
- \( \overline{S}(x, y) = 0 \) signifies that statement "\( x S y \)" is *logically undetermined*, i.e. could be either true or false;
- \( \overline{S}(x, y) < 0 \) signifies that assertion "\( x S y \)" is *more false than true*. There is only a minority of the criteria which warrants the truth of the outranking. This is equivalent to saying that a sufficient majority of criteria warrants the truth of the negation of the outranking;
- \( \overline{S}(x, y) = -1 \) signifies that assertion "\( x S y \)" is *certainly false*.

**Definition 1.3.** The set \( X \) associated to a bipolar-valued characterisation \( \overline{S} \) of the outranking relation \( S \in X \times X \) is called a bipolar-valued outranking *digraph*, denoted \( \overline{G}(X, \overline{S}) \).

From the truth-denotation semantics of a bipolar-valued characterisation it results that we can recover the crisp outranking \( S \) characterised via \( \overline{S} \) as the set of pairs \( \langle x, y \rangle \) such that \( \overline{S}(x, y) > 0 \). We write \( G(X, S) \) the corresponding so-called *strict 0-cut crisp outranking digraph* associated to \( \overline{G}(X, \overline{S}) \).

**Example 1** Consider a set \( X_1 = \{a, b, c, d, e\} \) of decision alternatives randomly evaluated on a coherent family \( F_1 = \{1, \ldots, 5\} \) of criteria of equal significance (see left part of Table 1.1). On each criterion we observe a rational preference scale from 0 to 1 with an indifference threshold of 0.1, a preference threshold of 0.2, a weak veto threshold of 0.6, and a veto threshold of 0.8. Based on the performances of the five alternatives on each criterion, we compute the bipolar-valued outranking relation \( \overline{S}_1 \) shown in the right part of Table 1.1. The strict 0-cut crisp digraph \( G_1(X_1, S_1) \) associated to the bipolar-valued outranking digraph \( \overline{G}_1(X_1, \overline{S}_1) \) is shown in Figure 1.1. Note the dotted arc from alternative e to d which represents an undetermined outranking. This situation is not expressible in a standard Boolean-valued
The RuBy best choice method

<table>
<thead>
<tr>
<th>decision objects</th>
<th>coherent family of criteria</th>
<th>$a$</th>
<th>$b$</th>
<th>$\tilde{S}$</th>
<th>$c$</th>
<th>$d$</th>
<th>$e$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a$</td>
<td>0.52 0.82 0.07 1.00 0.04</td>
<td>1.0</td>
<td>-0.2</td>
<td>-1.0 0.6 0.4</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$b$</td>
<td>0.96 0.27 0.43 0.83 0.32</td>
<td>0.4</td>
<td>1.0</td>
<td>0.2 0.2 0.4</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$c$</td>
<td>0.85 0.31 0.61 0.41 0.98</td>
<td>0.2</td>
<td>0.4</td>
<td>1.0 0.4 0.6</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$d$</td>
<td>0.30 0.60 0.74 0.02 0.02</td>
<td>-1.0</td>
<td>-1.0</td>
<td>-1.0 1.0 -1.0</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$e$</td>
<td>0.18 0.11 0.23 0.94 0.63</td>
<td>0.2</td>
<td>0.2</td>
<td>-0.4 0.0 1.0</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table 1.1. Example 1: Random performance table and bipolar-valued outranking relation.

Fig. 1.1. Example 1: Associated strict 0-cut digraph and undetermined arc characterisation of the outranking. Consequently, the (‘positive’) negation of the general $\tilde{S}$ relation is not identical to the complement of $S$ in $X \times X$.

Let us finish this subsection by introducing some concepts which are used in the sequel. The order $n$ of the digraph $\tilde{G}(X, S)$ is given by the cardinality of $X$, whereas the size $p$ of $\tilde{G}$ is given by the cardinality of $S$. As $X$ is a finite set of $n$ alternatives, the size $p$ of the digraph $\tilde{G}$ is also finite. A digraph $\tilde{G}(X, S)$ of order $n$ is said to be connected if the symmetric and transitive closure of $G(X, S)$ equals $K_n$.

A path of order $m \leq n$ in $G(X, S)$ is a sequence $(x_i)_{i=1}^m$ of alternatives of $X$ such that $\tilde{S}(x_i, x_{i+1}) \geq 0$, $\forall i \in \{1, \ldots, m - 1\}$. A circuit of order $m \leq n$ is a path of order $m$ such that $\tilde{S}(x_m, x_1) \geq 0$.

**Definition 1.4.** An odd chordless circuit $(x_i)_{i=1}^m$ is a circuit of odd order $m$ such that $\tilde{S}(x_i, x_{i+1}) \geq 0$, $\forall i \in \{1, \ldots, m - 1\}$, $\tilde{S}(x_m, x_1) \geq 0$ and $\tilde{S}(x_i, x_j) < 0$ otherwise.

Following a result by Bouyssou (Bou96), it appears that, apart from certainly being reflexive, the bipolar-valued outranking digraphs do not necessarily possess any special relational properties such as transitivity or complete comparability. Indeed, with a sufficient number of criteria, it is always possible to define an ad hoc performance table such that the associated crisp 0-cut outranking digraph renders any given reflexive binary relation. This rather positive result from a methodological point of view – the outranking based methodology is universal – bears however a negative
algorithmic consequence. Solving the best choice problem based on a general bipolar-valued outranking digraph becomes a non trivial algorithmic problem in case of non-transitive and partial outrankings, as we will show in the next section.

Nevertheless, before studying this problem in detail, we need to introduce the concept of choice in a bipolar-valued outranking digraph.

1.3 On choices and kernels in bipolar-valued outranking digraphs

A choice in a given bipolar-valued outranking digraph is a non-empty subset of decision objects.

Definition 1.5.

1. A choice \( Y \) in \( \tilde{G}(X, \tilde{S}) \) is said to be outranking (resp. outranked) if and only if \( x \notin Y \Rightarrow \exists y \in Y : \tilde{S}(y, x) > 0 \) (resp. \( \tilde{S}(x, y) > 0 \));
2. A choice \( Y \) is said to be independent (resp. strictly independent) if and only if for all \( x \neq y \) in \( Y \) we have \( \tilde{S}(x, y) \leq 0 \) (resp. \( \tilde{S}(x, y) < 0 \));
3. An outranking (resp. outranked) and independent choice is called an outranking (resp. outranked) kernel;
4. An outranking (resp. outranked) and strictly independent choice is called a determined outranking (resp. outranked) kernel.

Example 1 (continued) In the strict 0-cut crisp digraph \( G_1 \) (see Figure 1.1) we can observe two determined outranking kernels, namely the singletons \( \{b\} \) and \( \{c\} \). The digraph also contains one outranked kernel, namely the pair \( \{d,e\} \). Note that alternatives \( d \) and \( e \) are independent (but not strictly independent) from each other.

A choice \( Y \) in \( \tilde{G}(X, \tilde{S}) \) may be characterised with the help of bipolar-valued membership assertions \( \tilde{Y} : X \to \mathcal{L} \), denoting the credibility of the fact that \( x \in Y \) or not, for all \( x \in X \). \( \tilde{Y} \) is called a bipolar-valued characterisation of \( Y \), or for short a bipolar-valued choice in \( \tilde{G}(X, \tilde{S}) \).

Based on the truth-denotation semantics of the bipolar-valued characterisation domain \( \mathcal{L} \) (see Subsection 1.1), we obtain the following properties:

- \( \tilde{Y}(x) = +1 \) signifies that assertion “\( x \in Y \)” is certainly true;
- \( \tilde{Y}(x) > 0 \) signifies that assertion “\( x \in Y \)” is more true than false;
- \( \tilde{Y}(x) = 0 \) signifies that assertion “\( x \in Y \)” is logically undetermined, i.e. could be either true or false;
- \( \tilde{Y}(x) < 0 \) signifies that assertion “\( x \in Y \)” is more false than true;
- \( \tilde{Y}(x) = -1 \) signifies that assertion “\( x \in Y \)” is certainly false. Equivalently, one can say that assertion \( x \notin Y \) is certainly true.
In the following paragraphs, we recall useful results from (BPR06). They allow us to establish a link between the classical graph-theoretic and algebraic representations of kernels (via their bipolar-valued characterisations).

**Proposition 1.6.** (BPR06) The outranking (resp. outranked) kernels of \( \tilde{G}(X, \tilde{S}) \) are among the bipolar-valued choices \( \tilde{Y} \) satisfying the respective following bipolar-valued kernel equation systems:

\[
\max_{y \neq x} \left[ \min \left( \tilde{Y}(y), \tilde{S}(y, x) \right) \right] = -\tilde{Y}(x), \quad \text{for all } x \in X; \tag{1.3}
\]

\[
\max_{y \neq x} \left[ \min \left( \tilde{S}(x, y), \tilde{Y}(y) \right) \right] = -\tilde{Y}(x), \quad \text{for all } x \in X. \tag{1.4}
\]

Let \( \mathcal{Y}^+ \) and \( \mathcal{Y}^- \) denote the set of bipolar-valued choices verifying the respective kernel equation systems (1.3) and (1.4) above. Let \( Y_1 \) and \( Y_2 \) be two elements of \( \mathcal{Y}^+ \) (or \( \mathcal{Y}^- \)). \( Y_1 \) is said to be at least as sharp as \( Y_2 \) (denoted \( Y_2 \preceq Y_1 \)) if and only if for all \( x \) in \( X \) either \( Y_1(x) \leq Y_2(x) \leq 0 \) or \( 0 \leq Y_2(x) \leq Y_1(x) \). The \( \preceq \) relation defines a partial order (asymmetric and transitive) (Bis97). If \( \tilde{Y}(x) \neq 0 \) for each \( x \) in \( X \), \( \tilde{Y} \) is called a determined bipolar-valued choice.

**Theorem 1.7.**

1. There exists a one-to-one correspondence between the maximal sharp determined choices in \( \mathcal{Y}^+ \) (resp. \( \mathcal{Y}^- \)) and the determined outranking (resp. outranked) kernels in \( \tilde{G} \).
2. Each maximal sharp choice in \( \mathcal{Y}^+ \) (resp. \( \mathcal{Y}^- \)) characterises an outranking (resp. outranked) kernel in \( \tilde{G} \).

**Proof.** The first result, specialised to determined choices, is proved in (BPR06, theorem 1). The second one results directly from the kernel equation systems of Proposition 1.6. \( \square \)

The maximal sharp choices in \( \mathcal{Y}^+ \) (resp. \( \mathcal{Y}^- \)) deliver thus outranking (resp. outranked) kernel characterisations. It is worthwhile noting that not all partially determined outranking or outranked kernels admit, however, a maximal sharp solution. In the case of the presence of an odd cordless circuit for instance, it may happen that no maximal sharp solution is defined. Let us furthermore notice, that it may also happen that neither \( \mathcal{Y}^+ \) nor \( \mathcal{Y}^- \) contain any determined, or even partially determined, choices at all. For example it suffices to take \( \tilde{G} \) as an odd cordless circuit (see Definition 1.4).
Example 1 (continued) Recall that the strict 0-cut outranking digraph $G_1$ contains two outranking kernels and one outranked kernel. The bipolar-valued characterisations of these kernels are represented in Table 1.2. The outranking kernel $\{c\}$ looks more determined than $\{b\}$, and is therefore the more credible instance. Indeed, one can easily verify that the degrees of logical determination of the membership assertions for $\{c\}$ are higher than those for $\{b\}$ (recall Definition 1.2). Concerning the outranked kernel $\{d,e\}$, it is worthwhile noting that alternative $d$ belongs to it with certainty. The belonging of alternative $e$ to this kernel, however, depends on the undetermined situation $dS_e$. In view of a progressive approach, if this outranking becomes more true than false at a later stage, then $e$ can be dropped from the outranking kernel without any regrets. On the opposite, if the outranking becomes more false than true, then $e$ remains part of the thus completely determined kernel $\{d,e\}$.

In the past, the authors have promoted the most determined outranking kernel in a bipolar-valued outranking digraph $\tilde{G}$ as the potential candidate for a best choice recommendation.

Example 1 (continued) The reader can indeed easily verify in Table 1.1 that alternative $c$ is performing better than alternative $b$. Alternative $a$ has very contrasted performances and $d$ definitely shows the worst performances.

However, recent well founded criticisms against the capacity of the outranking kernel concept to render in general a satisfactory and convincing best choice recommendation lead us to refine our position. We therefore revisit the foundations of the RuBY BC decision aiding methodology in the next section. As we will see in Section 3, this study will lead us to the concept of hyperkernel in the outranking digraph as a BC recommendation. Note however that in most of the cases the outranking kernel is an appropriate answer to the BC decision problem in the context of an outranking relation (as shown experimentally in Section 4.3).

2 Foundations of the RuBY BC decision aiding methodology

We start this section by revisiting the BC problematics in order to identify the type of pragmatic decision aid we are addressing. A comparison with
the classical ELECTRE approach will underline similarities and differences with the RuBY approach. Finally, we present new pragmatic foundations for the BC decision aiding methodology.

2.1 The best choice problematique

From a classical OR point of view, the BC decision problem is the search for one best or optimal alternative. From a decision aid point of view, however, the assistance we may offer the decision maker depends on the operational objectives and characteristics of the underlying problem and the decision aiding process.

Operational objective:
Depending on the pragmatic goals of the decision process, the BC problem may be classified as the:
1. Search for a one best alternative (1-BCP), or the
2. Search for a set of \( k \) simultaneously best alternatives (\( k > 1 \)) (\( k \)-BCP).

In the second situation, the set of the \( k \) retained alternatives is considered to be a BC as a whole. This means, in particular, that each of the \( k \) alternatives taken individually may not be a BC. In this work we focus on the 1-BCP.

Type of decision aiding process:
From a decision aid point of view, the BC decision aid methodology is the study which analyses the principles and procedures of resolution of a particular version of the BC problem, whereas a BC decision aiding method is one particular approach to cope and to solve a given BC problem. A best choice recommendation (denoted BCR) is the output of BC decision aiding method.

As we will see in Subsection 2.1, such a BCR may be an intermediate solution to the BC problem. It is a set of potential best alternatives which will be refined in a future step via interactions with the DM. Therefore, we have to carefully distinguish between the current BCR of the decision aiding process and a final BC solution, which is the BCR proposed in the final step of a decision aiding process.

With this in mind, we can distinguish between two kinds of BC decision aiding approaches depending on the nature of the underlying decision problem:
1. A BC problem which requires the ultimate BCR in a single decision aiding step, and
2. A BC problem which allows to progressively uncover the ultimate BCR through the implementation of an iterative, progressive multiple step decision aiding process.
As already mentioned, in the second type of situation, a current BCR is generally not an ultimate solution to the given BC problem, but contains all potential candidates for such an ultimate BCR.

Solving the BC problem in a single step means finding a so-called *optimal alternative*, a decision object which is considered strictly better or at least equivalent to all other potential alternatives. The search for such an alternative is therefore based on the following implicit existence axiom (Roy81):

There exists at least one decision alternative which, with sufficient time and means, may be objectively proved as being optimal whilst remaining neutral in relation to the decision process.

In (Roy81), three fundamental conditions are listed to give sense to the concept of optimality and guarantee the existence of at least one optimal alternative.

1. The set \( X \) of alternatives must be given beforehand and cannot be changed during the decision process.
2. The alternatives of \( X \) must all be comparable, and,
3. These comparisons must be transitive.

This last condition is very strong and unrealistic when involving multiple preference dimensions in the present context of pairwise comparison outranking methods.

We focus in the sequel on the progressive BC decision aiding approach in the tradition of the French decision aid school.

### 2.2 The Electre best choice decision aiding methodology

The progressive 1-BC approach is extensively discussed in the context of multiple criteria decision aid in (RB93). Its ultimate goal is to select a single *best* alternative. The decision aiding process is therefore turned towards the elicitation of potential best choice candidates, i.e. a BCR which is as restricted as possible, in each step of the decision aiding process. This BCR is meant either to uncover the unique best alternative — the ultimate step of the decision aiding process —, or to enlighten the DM on the next step which has to be taken in the ongoing process.

Compared to what was said in Subsection 2.1 above concerning the single step BCP and the optimisation problem, it is important to note here that the set of alternatives \( X \) could be revisable or transitory. Furthermore, nothing is said about the transitivity of the relation which allows to compare all the alternatives in \( X \). The philosophy of the progressive BC decision aid methodology is therefore not to force the determination of a single best alternative at any cost at a given step of the decision process.
As Roy explains (Roy85), it is more important to make sure that the non-retained alternatives of the BCR have been left out for well-founded reasons, acknowledged and approved by the DM. Therefore, instead of forcing the decision aid procedure to elicit a single best alternative, it is preferable to obtain a choice \( Y \) of potential best alternatives, as long as this choice can be plainly justified on the basis of the currently available preferential information. This means from a methodological point of view that, at a given state of the decision process, all available preferential arguments for rejecting a maximum of alternatives from the best choice recommendation must be exploited.

Starting from this methodological position, Roy defines two pragmatic principles for the progressive BCP. A choice \( Y \) in \( X \) is a BCR if it satisfies the following two requirements:

\[ P_1: \text{Each alternative which is not selected in } Y \text{ must be outranked by at least one alternative of } Y; \]
\[ P_2: \text{The number of retained alternatives in the choice } Y \text{ must be as small as possible.} \]

The first principle counterbalances the second one. Indeed, on the one hand, \( P_1 \) tends to keep the cardinality of the BCR high enough to guarantee that no potential best choice is missed out. On the other hand, \( P_2 \) tends to keep its cardinality as small as possible in order to focus on the unique best choice. In particular, the cardinality argument in the second requirement signifies that no alternative of a BCR can be removed without violating the first principle \( P_1 \).

The immediate consequence of these principles is that the alternatives retained in a BCR are indeed potential best choice candidates. A further analysis, restricted to the actual choice \( Y \) of recommended potential best alternatives, may eventually reduce the number of decision candidates, or – the ideal case – let appear in an ultimate step of the decision aid process the actual unique best choice. It may happen, however, when \( Y = X \) or \( Y \) is empty for instance, that no BCR can be established in the given configuration of the decision process.

In the context of the ELECTRE methods, Roy (Roy85) proposes to use the concept of *outranking kernel*, i.e. the independent and outranking choice, as BCR. One can indeed easily check that this BCR verifies both \( P_1 \) and \( P_2 \). According to Roy, this BCR has to be unique. Nevertheless, the existence of such a unique outranking kernel is only guaranteed when the digraph does not contain any circuits at all (Ber70). To avoid a possible emptiness or a possible multiplicity of outranking kernels, Roy proposes either, to reduce some or all of the maximal circuits of the digraph to equivalence classes (ELECTRE I), or, to increase the discrimination between certain alternatives in order to break these circuits (ELECTRE IS). This hopefully minimal perturbation of the initial outranking digraph is based upon subtle robust-
ness considerations concerning the data of the bipolar-valued outranking relation $\tilde{S}$ (RB93).

In this work we do not follow the same approach. Rathermore we revisit the very foundations of a progressive BC decision aid methodology in order to discover how the concept of outranking kernel of a bipolar-valued outranking digraph may be used in order to deliver satisfactory BCRs without having to modify the bipolar-valued characterisation of the outranking relation.

2.3 New pragmatic foundations for a progressive BC decision aid methodology

In this subsection we propose five pragmatic principles that a progressive BC decision aid method should follow. They stem from careful considerations of existing approaches and their drawbacks.

$P_1$: Rejection for well motivated reasons

Each non-retained alternative for a BCR must be rejected for well motivated reasons in order not to miss out on any potential best alternative.

A similar formulation of this pragmatic principle is to say that each non-retained alternative for the BCR must be worse than at least one alternative retained in the BCR.

$P_2$: Minimal size

The number of alternatives retained in a BCR should be as small as possible.

This requirement is obvious when recalling that the goal is to find a unique best alternative. In light of this, a BCR containing a single element is the ideal case.

$P_3$: Efficient and informative refinement

Each step of the progressive decision aid must deliver an efficient and informative refinement of the previous recommendation.

This pragmatic principle of efficiency is necessary in order to avoid unnecessary iterations in the progressive approach to resolve the 1-BC problem, and to prevent the DM from having to face obvious information which would lead him to draw erroneous conclusions. At each step of the decision aiding process, the delivered BCR must focus on new and previously unknown preference statements, such that the progressive decision aid converges to a unique best alternative as quickly and efficiently as possible.

To some extent, this requirement is closely linked to the pragmatic principle $P_2$. Nevertheless, in the sequel we will present the very importance of this principle.
The RuBy best choice method

$P_4$: Effective BC recommendation

The BCR should not correspond simultaneously to a best and a worst choice.

This principle implies that if all decision alternatives are considered as equivalent, no effective BCR can be done.

$P_5$: Maximal credibility

The BCR must be as credible as possible with respect to the preferential knowledge available in the current step of the decision aiding process.

As the credibility degrees in the bipolar-valued outranking digraph represent the more or less overall concordance or consensus of the criteria for supporting a given outranking situation, it seems natural that in the case of several potential best choices, we recommend the one(s) with the highest determinateness of the membership assertions.

The first two principles are identical to those proposed by Roy (see Section 2). As we show in the next section, they are generally not sufficient to guarantee a satisfactory BCR. The three new principles $P_3$, $P_4$, and $P_5$ will show their operational value when translated into properties in the bipolar-valued outranking digraph of the BCRs.

Definition 2.1. A BCR which verifies the five pragmatic principles above is called a RuBy best choice recommendation.

The task which appears now is to see to which properties in the outranking digraph these five pragmatic principles lead, and which graph-theory related object these properties characterise. The following section deals with this translation and introduces the concepts of hyperindependence and hyperkernel.

3 Tackling the BCP in the bipolar-valued outranking digraph

This section deals with the consequences the five pragmatic principles, introduced in Subsection 2.3, have on the formal characterisation of a RuBy BCR. For each of the principles, we detail its operational meaning, its methodological consequence and a translation into a formal property using the vocabulary and concepts of the bipolar-valued outranking digraph. We also show that the formal properties derived from the above pragmatic principles lead to a unique concept – namely the hyperkernel – which will represent the BCR in an outranking digraph. As this object directly follows by construction from the pragmatic principles, it is also the very answer to the progressive BC problem.
Beforehand, we start this discussion by mentioning some particular cases where there exist obvious RuBY bcrs. When the 0-cut crisp outranking relation S has certain structural properties, the solution to the BCP may indeed become obvious.

- If S is a total order for instance, the best choice corresponds to the unique optimal alternative.
- In the case of a partial order (with incomparabilities), all initial alternative(s) of the outranking relation are potential best choice candidates and must be retained for a further decision aid step.
- If S is a total preorder (with indifferences or equivalence classes), the best choice recommendation is given by the equivalent best alternatives.
- In the case of a partial preorder (incomparabilities and indifferences), the alternatives of all initial equivalence classes of the outranking relation must be retained for a further decision aid step.

However, as already mentioned earlier, the 0-cut crisp outranking digraphs we may obtain from our bipolar-valued outranking digraphs are in general not of these particular types. Therefore, it is necessary to find a procedure which computes a bcr verifying the five RuBY principles with respect to any possible reflexive binary relation.

Throughout this section, we will illustrate our purpose with the help of the following didactic example\(^2\).

**Example 2** Let \( \overline{G}_2(X_2, \overline{S}_2) \) be a bipolar-valued outranking digraph, where \( X_2 = \{a, b, c, d, e\} \) and \( \overline{S}_2 \) is given in table 3.1. The associated strict 0-cut crisp digraph \( G_2(X, S) \) is represented in figure 3.1. The negation of the S relation is thus again not identical to the complement of S in \( X \times X \).

<table>
<thead>
<tr>
<th>( \overline{S}_2 )</th>
<th>a</th>
<th>b</th>
<th>c</th>
<th>d</th>
<th>e</th>
</tr>
</thead>
<tbody>
<tr>
<td>a</td>
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<td>0.2</td>
<td>-1.0</td>
<td>-0.7</td>
<td>-0.8</td>
</tr>
<tr>
<td>b</td>
<td>-0.6</td>
<td>1.0</td>
<td>0.8</td>
<td>1.0</td>
<td>0.0</td>
</tr>
<tr>
<td>c</td>
<td>-1.0</td>
<td>-1.0</td>
<td>1.0</td>
<td>0.2</td>
<td>0.8</td>
</tr>
<tr>
<td>d</td>
<td>0.6</td>
<td>-0.6</td>
<td>-1.0</td>
<td>1.0</td>
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<td>e</td>
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<td>-0.8</td>
<td>-0.4</td>
<td>-0.6</td>
<td>1.0</td>
</tr>
</tbody>
</table>

**Table 3.1.** Example 2: the bipolar-valued outranking relation

Let us now analyse the previously mentioned pragmatic principles one by one and present their translations in terms of the concepts presented in Section 1.

\(^2\) B.Roy, 2005, private communication.
3.1 Rejection for well motivated reasons (pragmatic principle: $P_1$)

In terms of the bipolar-valued outranking relation, pragmatic principle $P_1$ amounts to saying that each non-retained alternative should be outranked by at least one alternative of the BCR.

$R_1$: Outranking

A RuBy BCR must be an outranking choice in $\tilde{G}(X, \tilde{Y})$.

Example 2 (continued) Many choices in $\tilde{G}_2$ verify this property. The choices $\{a, b, c\}, \{b, c, d\}$, as well as $\{a, b, c\}$ for instance, are all outranking choices in this reference example.

3.2 Minimal size and efficient and informative refinement (pragmatic principles: $P_2$ and $P_3$)

In this subsection we show that these two pragmatic principles are closely linked. To rewrite principle $P_2$ – Minimal size – in the present context, we first need to define some graph-theory related concepts.

Definition 3.1.

1. The outranking neighbourhood $\Gamma^+(x)$ of a node (or equivalently an alternative) $x$ of $X$ is the union of $x$ and the set of alternatives which are outranked by $x$;
2. The outranking neighbourhood $\Gamma^+(Y)$ of a choice $Y$ is the union of the outranking neighbourhoods of the alternatives of $Y$;
3. The private outranking neighbourhood $\Gamma^+_Y(x)$ of an alternative $x$ in a choice $Y$ is the set $\Gamma^+(x) \setminus \Gamma^+(Y \setminus \{x\})$.

For a given alternative $x$ of a choice $Y$, the set $\Gamma^+_Y(x)$ represents the personal contribution of $x$ to the outranking quality of $Y$. If the private outranking neighbourhood of $x$ in $Y$ is empty, this means that, when $x$ is dropped from this choice, $Y$ still remains an outranking choice. From this observation one can derive the following definition.
Definition 3.2. A choice $Y$ is said to be irredundant if all the alternatives of $Y$ have non-empty private neighbourhoods.

Note that irredundancy is a directed concept related to either an outranking or an outranked property. In the case of an irredundant outranking choice $Y$ for instance, all the alternatives of $Y$ have non-empty private outranking neighbourhoods. The formal counterpart of the minimal size principle is therefore that of irredundancy of the choice.

Example 2 (continued) Among all the possible choices of the reference example, only few are outranking as well as irredundant: $\{a,b,e\}$, $\{b,c,d\}$, $\{b,e,d\}$, and $\{a,c\}$. The choice $\{a,b,c\}$ listed in the context of principle $R_1$ is not an irredundant outranking choice, as alternative $b$ has an empty private neighbourhood in this choice.

As mentioned previously in Subsection 2.2, principles $R_1$ and $R_2$ were already introduced in (Roy81; RB93) in the context of methodological studies on the BCP where they lead to the concept of outranking kernel. One should notice here that these two principles generate all the irredundant outranking choices of the digraph (and not solely the outranking kernels).

Let us now switch to the principle of efficient and informative refinement $P_3$. Considering the progressive approach that we are promoting here, the primary objective of this principle is to avoid that the DM has to face obvious and confirmed outranking arguments in the given BCR without a further analysis. We therefore require that a BCR $Y$ in an outranking digraph $G(X,S)$ should be such that the digraph restricted to the nodes of a BCR cannot contain a BCR obvious at that stage of the progressive decision process.

Consequently, at each step of the decision aiding process, the BCR delivered must focus on new and previously undetermined or unknown preference statements. Let us illustrate this with a short example.

Example 3 Consider the problem shown on the strict median cut digraph represented in figure 3.2. Both highlighted choices $Y_1 = \{a,b\}$ and $Y_2 = \{a,d,e,f,\ldots,z\}$ verify the pragmatic principles $P_1$ and $P_2$, as outranking irredundant choices. One would be tempted to prefer $Y_1$ to $Y_2$ because of its lower cardinality. Nevertheless, $Y_1$ contains information which is already confirmed at this stage of the progressive search; namely that the statement “$a$ outranks $b$” is true. In the case of the choice $Y_2$, the next step of the search will focus on alternatives which were incomparable at the previous analysis. This lack of information will probably be solved in the next step of the process. If this further step focuses on the choice $Y_1$, then it is quite difficult to imagine that the DM will be able to forget about the already confirmed truth of the statement “$a$ outranks $b$”. He will most certainly consider $a$ as the best choice, which might however not be the best decision alternative, as
The best choice method

Fig. 3.2. Example 3: An unstable \( \{a, b\} \) and a stable \( \{d, e, f, \ldots, z\} \) choice.

nothing is known at this stage of the analysis about the relations between \( a \) and \( \{d, e, f, \ldots, z\} \).

Recall that the pragmatic principles \( P_1 \) and \( P_2 \) generate the outranking irredundant choices. In view of the previous observations we define the concept of BC-stability as follows:

**Definition 3.3.** An outranking (resp. outranked) choice \( Y \) in \( \tilde{G}(X, \tilde{V}) \) is said to be BC-stable if and only if the induced subgraph \( \tilde{G}_Y(Y, \tilde{S}|_Y) \) does not contain any outranking (resp. outranked) irredundant choice.

The concept of outranking (resp. outranked) kernel (see Definition 1.5) in an outranking digraph verifies the property of BC-stability. Nevertheless, as already mentioned in Subsection 2.2, the existence of an outranking kernel is not guaranteed in an outranking digraph. This is shown in the following property.

**Property 3.4.** If a digraph \( \tilde{G}(X, \tilde{S}) \) has no outranking (resp. outranked) kernel, it contains a chordless circuit of odd order.

**Proof.** This property represents the contraposition of Richardson’s general result: If a digraph contains no chordless circuit of odd order, then it has an outranking (resp. outranked) kernel (see Ric53). \( \square \)

As shown in Property 3.4, the outranking kernel is well adapted as potential BCR in case the outranking digraph does not contain any chordless circuit of odd order (Definition 1.4). Consider now the case where the BCR, resulting from pragmatic principles \( P_1 \) and \( P_2 \), is a chordless circuit \( Y = \{a, b, c\} \) of order 3 such that \( a S b, b S c \) and \( c S a \). Such a BCR is clearly neither a kernel nor is it BC-stable because there are three possible further BCRs which can be put forward on the digraph restricted to the elements of \( Y \) (namely \( \{a, b\}, \{b, c\} \) and \( \{a, c\} \)). Nevertheless, it is an interesting recommendation because it presents 3 alternatives to the DM which don’t contain
obvious information on the possible unique BC at this step of the progressive search. In fact, \(a, b\) and \(c\) are considered to be equivalent potential best alternatives in the current stage of the decision process.

In view of the previous observations, it seems clear that neither the concepts of stability and irredundancy nor that of outranking kernel are adapted for the search of a BCR in case of a general outranking digraph. Firstly potentially interesting BCRs are left out and, secondly, nothing guarantees the existence of a kernel in an outranking digraph. In order to overcome these problems, we introduce the concept of \textit{hyperindependence}, an extension of the independence property discussed in Section 1.3:

\textbf{Definition 3.5.} A choice \(Y\) in \(\tilde{G}\) is said to be (strictly) hyperindependent if it consists of chordless circuits of odd order \(p \geq 1\) which are (strictly) independent of each other.

Note that in Definition 3.5 above, singletons are assimilated to chordless circuits of (odd) order 1.

Pragmatic principles \(P_2\) and \(P_3\) can now be translated into the formal property \(R_2:\)

\(R_2:\) \textbf{Hyperindependence}

A \textsc{RuBy} BCR must be a hyperindependent choice in \(\tilde{G}(X, \tilde{Y})\).

As a direct consequence, we define the concept of \textit{hyperkernel}.

\textbf{Definition 3.6.} An outranking (resp. outranked) hyperindependent (resp. strictly hyperindependent) choice is called an outranking (resp. outranked) hyperkernel (resp. determined hyperkernel).

\textbf{Example 2 (continued)} Choice \(\{a, b, d, e\}\) (see Figure 3.1) is an outranking hyperkernel. The fact that the outranking relation between \(d\) and \(e\) is undetermined implies that choice is not strictly hyperindependent, and that it is not an outranking determined hyperkernel.

Note that in case the outranking digraph contains no chordless circuits of odd order 3 and more, the outranking kernels of the digraph deliver BCRs verifying the first two formal principles \(R_1\) and \(R_2\) (and indirectly the three pragmatic \textsc{RuBy} principles). In the general case however, the \textsc{RuBy} BCR will consist of a set of outranking hyperkernels of the digraph.

Last but not least, let us show on a short example that the concept of hyperkernel is appropriate for the BC problem we are dealing with.

\textbf{Example 4} Consider the outranking digraph of figure 3.3. Choice \(\{a, c, e\}\) is a BCR as an outranking kernel and choice \(\{b, d, e, f\}\) is an outranked hyperkernel. This second (bad) choice wouldn’t have been detected without the introduction of the concept of hyperindependence.
3.3 Effective outranking and maximal credibility (principles $\mathcal{P}_4$ and $\mathcal{P}_5$)

In order to translate principle $\mathcal{P}_4$ – Effective outranking – we have to define the concept of strict outranking. Recall that to an outranking (resp. outranked) choice $Y$ one can associate a bipolar-valued characterisation $\tilde{Y}^+$ (resp. $\tilde{Y}^-$). In order to appreciate if a choice $Y$ is an outranking or an outranked choice – or which of these bipolar-valued characterisations is the more determined –, we extend the concept of determinateness of propositional statements introduced earlier in definition 1.2 to the bipolar-valued characterisations of choices.

**Definition 3.7.** The determinateness $D(\tilde{Y})$ of the bipolar-valued characterisation $\tilde{Y}$ of a choice $Y$ is the average value of the determinateness $D(\tilde{Y}(x))$ for all $x$ in $X$.

We can now define the concept of strictness of a choice as follows:

**Definition 3.8.**

1. A null choice is a choice $Y$ which is outranking and outranked with a same credibility, i.e. for which the outranking and outranked bipolar-valued characterisations $\tilde{Y}^+$ and $\tilde{Y}^-$ have the same determinateness, i.e. $D(\tilde{Y}^+) = D(\tilde{Y}^-)$.

2. A strict outranking choice is a choice $Y$ for which its outranking bipolar-valued characterisation $\tilde{Y}^+$ has a higher determinateness than its outranked bipolar-valued characterisation $\tilde{Y}^-$, i.e. $D(\tilde{Y}^+) > D(\tilde{Y}^-)$.

3. A strict outranked choice is a choice $Y$ for which its outranked bipolar-valued characterisations $\tilde{Y}^-$ has a higher determinateness than its outranking bipolar-valued characterisation $\tilde{Y}^+$, i.e. $D(\tilde{Y}^-) > D(\tilde{Y}^+)$.

It now seems quite natural to translate the pragmatic principle of effectiveness $\mathcal{P}_4$ into a formal property $\mathcal{R}_3$ as follows:

**$\mathcal{R}_3$: Strictness**

A RuBy bcr is a strict choice in $\tilde{G}(X, \tilde{Y})$.

This concept allows to solve the problem raised by the following example.
Example 5 Consider the strict 0-cut outranking digraph represented on figure 3.4 (for the sake of simplicity we suppose that all the arcs which are drawn (resp. not drawn) represent a credibility of the outranking of 1 (resp. −1)). \{a\} and \{c\} are both irredundant outranking choices of the same maximal determinateness 1. However, one can easily see that alternative ‘a’ compares differently with ‘b’ than ‘c’ does. Alternative ‘c’ is clearly a null choice. If we now require the RuBY principles to be verified, only the choice \{a\} can be retained as a potential RuBY bcr.

An immediate consequence of the effectiveness principle is that a bipolar-valued outranking digraph, which is completely symmetrical – with equal credibility degrees for all $x S y$ and $y S x$ –, does not have any RuBY bcr. Every outranking choice will automatically be a null choice. Indeed, without any asymmetrical preferential statements, it is impossible to derive any preferential discriminations that would support a plausible strict outranking choice.

Finally, pragmatic principle $\mathcal{R}_5$ – Maximal credibility is again related to the determinateness of bipolar-valued choices (see Definition 3.7). In the case of multiple potential bcrs, we choose as RuBY bcr the most determined one, i.e. the one with the highest determinateness. Let $\hat{\mathcal{Y}}$ be the set of potential bcrs in $\hat{G}(Y, S)$ (verifying $\mathcal{R}_1, \mathcal{R}_2$ and $\mathcal{R}_3$).

$\mathcal{R}_4$: Maximal determinateness

A RuBY bcr is a choice in $\hat{G}(X, \hat{S})$ that belongs to the set

$$\hat{Y}^* = \{ \hat{Y}' \in \hat{\mathcal{Y}} | D(\hat{Y}') = \max_{\hat{Y} \in \hat{\mathcal{Y}}} D(\hat{Y}) \} \quad (3.1)$$

Example 1 (continued) Recall that in example 1 (see Subsection 1.2), we determined two outranking kernels which deliver potential bcrs (see Table 3.2). The determinateness of kernel \{c\} (0.36) is significantly higher than that of kernel \{b\} (0.20). Following formal principle $\mathcal{R}_4$, we recommend in this case the first solution, namely kernel \{c\}, as the RuBY bcr.

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3 This example is due to D. Bouyssou (RB93).
The RuBy best choice method

<table>
<thead>
<tr>
<th>$Y$</th>
<th>$a$</th>
<th>$b$</th>
<th>$c$</th>
<th>$d$</th>
<th>$e$</th>
<th>$D(Y)$</th>
<th>BCR</th>
</tr>
</thead>
<tbody>
<tr>
<td>$c$</td>
<td>-0.2</td>
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<td>-0.4</td>
<td>-0.4</td>
<td>0.36</td>
<td>RuBy</td>
</tr>
<tr>
<td>$b$</td>
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<td>0.2</td>
<td>-0.2</td>
<td>-0.2</td>
<td>-0.2</td>
<td>0.20</td>
<td>-</td>
</tr>
</tbody>
</table>

Table 3.2. Example 1: Illustration of the maximal credibility principle

In this section, we have presented the translation of the five RuBY principles into choice properties defined in the bipolar-valued outranking digraph. Detailed motivations for these principles have been given. They lead quite naturally to the new concept of outranking hyperkernel of an outranking digraph. We finish this section by a natural consequence of the previous considerations, namely the characterisation of the concept of RuBY bcr in a bipolar-valued outranking digraph.

**Theorem 3.9.** A choice in an outranking digraph $\tilde{G}(X, \tilde{S})$ is a RuBY bcr if and only if it verifies all four principles $R_1, R_2, R_3,$ and $R_4$, i.e. it is a maximally credible strict outranking hyperkernel.

The following section focuses on the construction of these hyperkernels and proposes a general algorithm for computation of the RuBY bcrs in any given bipolar-valued outranking digraph.

### 4 Computing the RuBy best choice recommendation

In this section, we start by presenting an approach which allows one to determine the hyperkernels of an outranking digraph before presenting some of their properties.

#### 4.1 Determination of the hyperkernels

We start by describing the procedure which will lead us to the characterisation of the hyperkernels of an outranking digraph $G(X, \tilde{S})$.

If $\tilde{G}$ contains chordless circuits of odd order, the original outranking digraph is modified into a digraph that we will call the chordless-odd-circuits-augmented (COCA) outranking digraph $\tilde{G}^c(X^c, \tilde{S}^c)$. Intuitively, the main idea is to “hide” the problematic circuits behind new nodes which are added to the digraph in a particular way. This may appear to be a criticable perturbation of the original information. Nevertheless, as we will see later, such a transformation does not affect the original problem but only helps to find more suitable solutions. The construction of the digraph $\tilde{G}^c(X^c, \tilde{S}^c)$ is described hereafter.
The procedure to obtain the COCA digraph $\tilde{G}^C$ is iterative. The initial digraph is written $\tilde{G}_0(X_0, \tilde{S}_0)$, and is equal to $\tilde{G}(X, \tilde{S})$. At step $i$, the set of nodes becomes $X_i = X_{i-1} \cup C_i$, where $C_i$ is a set of nodes representing the chordless circuits of odd order of $G_{i-1}(X_{i-1}, \tilde{S}_{i-1})$. These nodes are called hypernodes. The outranking relation $\tilde{S}_{i-1}$ is augmented by links between the nodes from $X_{i-1}$ and those from $C_i$ in the following way (the resulting relation is written $\tilde{S}_i$):

\begin{align*}
\forall C_k \in C_i & \left\{ \begin{array}{ll}
\tilde{S}_i(C_k, x) = \bigcup_{y \in C_k} \tilde{S}_{i-1}(y, x) & \forall x \in X_{i-1} \setminus C_k \\
\tilde{S}_i(C_k, x) = +1 & \forall x \in C_k
\end{array} \right., \quad (4.1)
\end{align*}

\begin{align*}
\forall x \in X_{i-1}, C_k \in C_i & \left\{ \begin{array}{ll}
\tilde{S}_i(x, C_k) = \bigcup_{y \in C_k} \tilde{S}_{i-1}(x, y) & \text{if } x \notin C_k \\
\tilde{S}_i(x, C_k) = +1 & \text{if } x \in C_k
\end{array} \right.. \quad (4.2)
\end{align*}

The iteration is stopped at step $r$ for which $|X_r| = |X_{r+1}|$. We then define $\tilde{G}^C(X^C, \tilde{S}^C)$ as the digraph $\tilde{G}_r(X_r, \tilde{S}_r)$. As the order of the original digraph $\tilde{G}$ is finite, the number of circuits it may contain is also finite. Therefore, the iteration is a finite process.

The iterative approach is necessary because of the fact that new chordless circuits of odd order may appear when new hypernodes are added to the digraph. Figure 4.1 presents such a case. First the chordless circuit $\{a, b, c\}$ of order 3 is detected. A new node labelled $\{a, b, c\}$ is added. Then the chordless circuit $\{\{a, b, c\}, d, e\}$ of order 3 is detected. Again, a new node labelled $\{\{a, b, c\}, d, e\}$ has to be added. No further odd chordless circuit can then be detected.

The outranking (resp. outranked) hyperkernels of $\tilde{G}(X, \tilde{S})$ are then determined by searching the classical outranking (resp. outranked) kernels of $\tilde{G}^C(X^C, \tilde{S}^C)$ (Bis97).

\footnote{For the sake of simplicity, an element $C_k$ of $C_i$ will represent a node of $X_i$ as well as a the set of nodes of $X_{i-1}$ representing the circuit $C_k$.}
4.2 Properties of the COCA outranking digraph

This extension of the digraph has two very important properties. First of all, the original outranking and outranked kernels of $G(X, \bar{S})$ can still be found in the associated COCA digraph $\tilde{G}^c(X^c, \bar{S}^c)$.

**Property 4.1.** The outranking (resp. outranked) kernels of $G(X, \bar{S})$ are also the outranking (resp. outranked) kernels of $G^c(X^c, \bar{S}^c)$.

**Proof.** Let us suppose that $G$ contains at least one odd chordless circuit. Let $Y$ be an outranking kernel of $G$ (the case of the outranked kernels can be treated similarly). We must prove that $Y$ is also an outranking kernel of $G^c$.

First, the elements of $Y$ are independent in $G$ and $G^c$ because no relation is added between elements of $X$ in $X^c$. Secondly, as $Y$ is a outranking choice in $\tilde{G}$, each element of $X \setminus Y$ is outranked by at least one element of $Y$. In particular, if $C_k$ is an odd chordless circuit of $X$, each node of $C_k$ is also outranked by at least one element of $Y$ (in $X$). Due to the special way $\bar{S}^c$ is built, the node representing $C_k$ in $X^c$ is also also outranked by at least one element of $Y$. $\square$

A second important property (shown in Property 4.3) is that there exists at least one outranking (resp. outranked) hyperkernel in $\tilde{G}^c$. To prove this property, we use the concept of inheritance.

**Definition 4.2.** In the COCA outranking digraph, a hypernode is said to inherit the outranking (outranked) characteristics of its corresponding odd chordless circuit.

An important property of this inheritance is that the outranking, as well as the outranked, neighbourhood of the odd chordless circuit are inherited by the hypernode. Besides, the circuit is outranked by and is outranking the hypernode (indifference) with a credibility of +1. Less formally, we could say that the hypernode is a representative of its corresponding circuit and that the circuit is somehow hidden behind it.

**Property 4.3.** The digraph $\tilde{G}^c(X^c, \bar{S}^c)$ contains at least one outranking (resp. outranked) hyperkernel.

**Proof.** This property directly follows from the construction principle of the COCA digraph (see Equations 4.1 and 4.2).

The main difficulty of determining kernels in an outranking digraph is related to the potential presence of odd chordless circuits (as shown in Property 3.4). Let us suppose that $G(X, \bar{S})$ contains no outranking kernel (a similar proof can be given for the outranked kernels). According to Property
3.4 This means that $\tilde{G}(X, \tilde{S})$ contains at least one odd chordless circuit. One can easily understand that if the structure of the digraph requires an element $x \in X$ of an odd chordless circuit $C_k$ to be in an irredundant outranking choice $Y$, due to the odd number of elements of that particular circuit, one of the two direct neighbours of $x$ in the circuit will also be added to $Y$. Consequently, $Y$ cannot be kernel in that situation.

Due to the particular construction of the associated COCA digraph $\tilde{G}^c$, there exists, for each odd chordless circuit, a hypernode which inherits its properties, and which is considered as indifferent to it. Consequently, each element of each odd chordless circuit in $G^c$ is outranked by, and is outranking, a hypernode. Furthermore, each of the hypernodes has the same outranking and outranked neighbourhoods as its corresponding odd chordless circuit.

Finally, the element $x$ of the odd chordless circuit $C_k$ will no longer be problematic in the construction of the outranking kernels of $\tilde{G}^c$ because there exists at least one hypernode which is equivalent to $x$, and which inherits from the outranking neighbourhoods of $C_k$. Consequently $C_k$ (as a hypernode) is added to $Y$ instead of $x$. □

We have shown in this subsection how the hyperkernels are determined by a modification of the outranking digraph. We have also proven that this modification does not affect the original information contained in the outranking relation. The following subsection details the RuBY proposal for a BC procedure based on the observations of this article.

4.3 The RuBY bcr algorithm

We summarise in this last subsection the overall algorithm which allows us to compute a RuBY bcr.

Algorithm

Input: $\tilde{G}(X, \tilde{S})$;

1. Construct the associated COCA digraph $\tilde{G}^c(X \cup C, \tilde{S}^c)$ (if necessary);
2. Extract all outranking and outranked hyperkernels from $\tilde{G}^c$ (or the kernels of $G$);
3. Eliminate the null choices (if necessary);
4. Rank the strict outranking hyperkernels (or kernels) by decreasing logical determinateness $D$;

Output: The strict outranking (hyper)kernel(s) with the highest strictly positive determinateness.

The first step of the RuBY bcr algorithm is by far the most difficult to tackle, as the number of odd chordless circuits in a bipolar outranking digraph might be huge. To study this operational difficulty, we have compiled
a sample of 1000 bipolar-valued outranking digraphs generated from random performances of 20 alternatives evaluated randomly on 7 to 20 criteria with random significance distributions and random thresholds. In nearly 98% of the sample, the time to compute the COCA digraph on a standard desktop computer is less than a second. In one case, we observe an execution time of around 30 seconds, a result clearly linked to the number of odd chordless circuits in the digraph.

<table>
<thead>
<tr>
<th>number of odd chordless circuits</th>
<th>#</th>
<th>rel. (in %)</th>
<th>cum. (in%)</th>
<th>histogram</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>735</td>
<td>73.57</td>
<td>73.57</td>
<td>***********************</td>
</tr>
<tr>
<td>1</td>
<td>116</td>
<td>11.61</td>
<td>85.19</td>
<td>****</td>
</tr>
<tr>
<td>2</td>
<td>65</td>
<td>6.51</td>
<td>91.69</td>
<td>**</td>
</tr>
<tr>
<td>3</td>
<td>25</td>
<td>2.50</td>
<td>94.19</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>16</td>
<td>1.60</td>
<td>95.80</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>11</td>
<td>1.10</td>
<td>96.90</td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>10</td>
<td>1.00</td>
<td>97.90</td>
<td></td>
</tr>
<tr>
<td>7</td>
<td>4</td>
<td>0.40</td>
<td>98.30</td>
<td></td>
</tr>
<tr>
<td>8</td>
<td>6</td>
<td>0.60</td>
<td>98.30</td>
<td></td>
</tr>
<tr>
<td>9</td>
<td>1</td>
<td>0.10</td>
<td>99.00</td>
<td></td>
</tr>
</tbody>
</table>

Table 4.1. Number of odd chordless circuits in random bipolar-valued outranking digraphs of order 20 (1000 observations)

In Table 4.1, we notice that nearly 75% of the sample digraphs don’t admit any odd chordless circuit at all. In 99% of the observations, we note less than 10 hypernodes added to the original outranking digraph.

The second step of the RuBy bcr algorithm concerns the extraction of hyperkernels from the COCA digraph. From a theoretical point of view, this step is well-known to be computationally difficult (Chv73). However, this difficulty is directly linked to the arc-density, i.e. the relative size of the digraph. Only very sparse digraphs, showing an arc-density lower than 10% in the range of digraph orders which are relevant for the BC decision aid problematique (10-30 alternatives), may present difficulties for the search of kernels. Figure 4.2 shows the histogram of the distribution of the arc-density on the sample of 1000 random outranking digraphs of order 20. The mean density observed here (82.6% with a standard deviation of 5.7%) is very high. Consequently, determining hyperkernels becomes a computing task practicable in more than reasonable time frames. Indeed, the mean execution time with its standard deviation for this step of the algorithm are around a thousandth of a second on a standard desktop computer.

Finally, eliminating the null hyperkernels and sorting the strict outranking hyperkernels in decreasing order of determinateness is linear in the order of the digraph and involves no computational difficulty at all.
Let us illustrate the RuBY bcr algorithm on the second example of this paper (see Section 3).

**Example 2 (continued)** The bipolar-valued outranking digraph of the example (see Figure 3.1) contains a chordless circuit of order 3, namely \{a, b, d\}. The original digraph \(\tilde{G}\) is extended to the digraph \(\tilde{G}^{C}\) which contains a hyper-node representing \{a, b, d\}. The corresponding outranking digraph admits an outranking kernel \{a, c\} and a hyperkernel \{\{a, b, d\}, e\} which is both outranking and outranked, but not with the same degree of determinateness as we may see in Table 4.3.

The hyperkernel \{\{a, b, d\}, e\} being more outranking than outranked, we stay here with two potential BCRs: The latter strict outranking hyperkernel \{\{a, b, d\}, e\} and the former strict outranking kernel \{a, c\}. The first one is significantly more determined (0.5 against 0.2) than the second one. The RuBY “best choice recommendation” therefore becomes \{a, b, d\}, e\}, where alternative e is in an undetermined situation. The ultimate belonging of alternative e to the BCR depends on the eventual reinforcement or the weakening of the credibility of an outranking situation between alternatives b and e revealed in some future decision aiding step.

<table>
<thead>
<tr>
<th>(\tilde{G}^{C})</th>
<th>a</th>
<th>b</th>
<th>c</th>
<th>d</th>
<th>e</th>
<th>{a, b, d}</th>
</tr>
</thead>
<tbody>
<tr>
<td>a</td>
<td>0.1</td>
<td>0.2</td>
<td>-1.0</td>
<td>-0.7</td>
<td>-0.8</td>
<td>1.0</td>
</tr>
<tr>
<td>b</td>
<td>-0.6</td>
<td>1.0</td>
<td>0.8</td>
<td>1.0</td>
<td>0.0</td>
<td>1.0</td>
</tr>
<tr>
<td>c</td>
<td>-1.0</td>
<td>-1.0</td>
<td>1.0</td>
<td>0.2</td>
<td>0.8</td>
<td>0.2</td>
</tr>
<tr>
<td>d</td>
<td>0.6</td>
<td>-0.6</td>
<td>-1.0</td>
<td>1.0</td>
<td>-0.4</td>
<td>1.0</td>
</tr>
<tr>
<td>e</td>
<td>1.0</td>
<td>-8</td>
<td>-0.4</td>
<td>0.6</td>
<td>1.0</td>
<td>-0.6</td>
</tr>
<tr>
<td>{a, b, d}</td>
<td>1.0</td>
<td>1.0</td>
<td>0.8</td>
<td>1.0</td>
<td>0.0</td>
<td>1.0</td>
</tr>
</tbody>
</table>

**Fig. 4.2.** Histogram of the arc density of a sample of 1000 COCA digraphs of order 20

**Table 4.2.** Example 2: the associated COCA relation
The RuBy best choice method

\[
\begin{array}{l|cccccc|c}
Y & \{a, b, d\} & a & b & c & d & e & D \\
\{\{a, b, d\}, e\}^+ & 0.6 & -0.6 & -0.6 & -0.6 & 0.0 & 0.5 & \\
\{a, c\} & -0.2 & 0.2 & -0.2 & 0.2 & -0.2 & 0.2 & \\
\{\{a, b, d\}, e\}^- & 0.0 & 0.0 & 0.0 & -0.6 & 0.0 & 0.6 & 0.2 \\
\end{array}
\]

Table 4.3. Example 2: the characteristic vectors of the outranking (+) and outranked (−) hyperkernels.

Let us conclude this paper with a reference to the Python digraphs module (Bis06), a freely downloadable Python implementation of computing resources, allowing one to work with bipolar-valued digraphs in general. A special procedure is provided which directly constructs the bipolar-valued outranking digraph and the corresponding RuBy bcr from a given performance table. All the examples in this paper, including the sample statistics, have been computed with this software package.

Concluding remarks

This article presents the foundations of the RuBy method for the best choice problem in multiple criteria decision aid. We started the discussion by introducing the concept of bipolar valuation of an outranking digraph, thereby providing degrees of confidence in the truth of a pairwise outranking relation on a set of decision alternatives. This bipolar valuation is centred around a situation of undeterminateness, allowing us to deal with incomplete and/or imprecise data comprehensively.

By basing the whole study on a set of 5 pragmatic principles, we have demonstrated the usefulness, and indeed necessity, of our approach of the best choice decision problem. The translation of these RuBy principles in properties of choices in a bipolar-valued outranking digraph leads us to the concept of hyperkernel, the latter being the very constituent for the RuBy best choice recommendation.

We close the discussion with the presentation of the RuBy best choice algorithm providing formal procedures – implemented in the Python programming language – for effectively computing a best choice recommendation in a practical case of multiple criteria decision aid.

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